The residue theorem from a numerical perspective

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Abstract

A short vignette illustrating Cauchy’s integral theorem using numerical integration

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In this very short vignette, I will use contour integration to evaluate

\[ \int_{x=-\infty}^{\infty} \frac{e^{ix}}{1+x^2} \, dx \tag{1} \]

using numerical methods. This document is part of the elliptic package Hankin (2006).

The residue theorem tells us that the integral of \( f(z) \) along any closed nonintersecting path is equal to \( 2\pi i \) times the sum of the residues inside it.

Take a semicircular path \( P \) from \(-R\) to \(+R\) along the real axis, then following a semicircle in the upper half plane, of radius \( R \) to close the loop. Now consider large \( R \). Then \( P \) encloses a pole at \( i \) [there is one at \(-i\) also, but this is outside \( P \), so irrelevant here] at which the residue is \(-i/2e\). Thus

\[ \oint_P f(z) \, dz = 2\pi i \cdot (-i/2e) = \pi/e \tag{2} \]

along \( P \); the contribution from the semicircle tends to zero as \( R \to \infty \); thus the integral along the real axis is the whole path integral, or \( \pi/e \).

We can now reproduce this result analytically. First, choose \( R \):

> R <- 400

And now define a path \( P \). First, the semicircle:

> u1 <- function(x){R*exp(pi*1i*x)}
> u1dash <- function(x){R*pi*1i*exp(pi*1i*x)}

and now the straight part along the real axis:

> u2 <- function(x){R*(2*x-1)}
> u2dash <- function(x){R*2}

And define the function:
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```r
> f <- function(z){exp(1i*z)/(1+z^2)}
Now carry out the path integral. I’ll do it explicitly, but note that the contribution from the
first integral should be small:

```r
> answer.approximate <-
+   integrate.contour(f,u1,u1dash) +
+   integrate.contour(f,u2,u2dash)
```
And compare with the analytical value:

```r
> answer.exact <- pi/exp(1)
> abs(answer.approximate - answer.exact)
[1] 6.244969e-07
```
Now try the same thing but integrating over a triangle instead of a semicircle, using integrate.segments(). Use a path $P'$ with base from $-R$ to $+R$ along the real axis, closed by
two straight segments, one from $+R$ to $iR$, the other from $iR$ to $-R$:

```r
> abs(integrate.segments(f,c(-R,R,1i*R))- answer.exact)
[1] 5.157772e-07
```
Observe how much better one can do by integrating over a big square instead:

```r
> abs(integrate.segments(f,c(-R,R,R+1i*R, -R+1i*R))- answer.exact)
[1] 2.319341e-08
```

0.1. Residue theorem

Function residue() is a wrapper that takes a function $f(z)$ and integrates $f(z)/(z - z_0)$
around a closed loop which encloses $z_0$. If $f(\cdot)$ is holomorphic within $C$, Cauchy’s residue
theorem states that

$$ \oint_C \frac{f(z)}{z - z_0} = f(z_0) \quad (3) $$

and we can test this numerically:

```r
> f <- function(z){sin(z)}
> numerical <- residue(f,z0=1,r=1)
> exact <- sin(1)
> abs(numerical-exact)
[1] 3.91766e-18
```
which is unreasonably accurate, IMO.

References


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